A HEAT METER OF FINITE THICKNESS

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The two-dimensional steady-state problem is solved for a heat meter of finite thickness. Corrections to the heat meter readings are estimated.

Heat meters of various constructions for obtaining experimental information on heat flux densities are being more and more widely used not only in research, but also to monitor and regulate processes in very diverse fields of industry [1].

Unfortunately, any device for measuring heat flux perturbs the thermal state of the object being investigated, and the recorded heat flux q_H differs from the unperturbed heat flux q_0:

$$q_0 = \alpha (T_0 - T).$$

Analysis of the thermal perturbation introduced by a heat meter and the derivation of the equation relating the recorded and unperturbed heat fluxes are of practical interest. Figure 1 shows a schematic diagram for the mathematical statement of the problem. The temperature of the medium is arbitrarily assumed equal to zero ($T_m = 0$). In the absence of the heat meter the wall close to the surface has a plane one-dimensional temperature distribution

$$T_1(z) = T_0 \left(1 + \frac{\alpha}{\lambda_1} z \right).$$

Upon insertion of the heat meter the temperature distribution of the wall is distorted and becomes two-dimensional. The heat-conduction equation has the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$$

with the boundary conditions

$$\lambda_2 \frac{\partial T_2}{\partial z} = \alpha T_2, \; z = -\delta, \; r < R, \tag{4}$$

$$\frac{\partial T_2}{\partial r} = 0, \; -\delta \leqslant z \leqslant 0, \; r = R, \tag{5}$$

$$\lambda_1 \frac{\partial T_1}{\partial z} = \alpha T_1, \; z = 0, \; r > R,$$

$$\lambda_2 \frac{\partial T_2}{\partial z} = \lambda_1 \frac{\partial T_1}{\partial z}, \; z = 0, \; r < R, \tag{7}$$

$$T_2 = T_1, \; z = 0, \; r < R.$$

The solution of such a steady-state problem presents certain difficulties. The main difficulty is that in the Or plane different boundary conditions (6)-(8) are specified along a ray. There is no general theory for such mixed problems [2, 3]. By using the Fourier method the following general expression is obtained for the temperature $T_2(z, r)$ of the heat meter:

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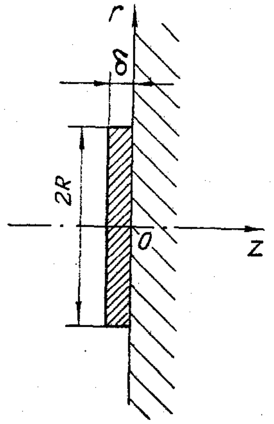

Fig. 1. Mathematical statement of problem.

$$T_2(z, r) = \sum_{n=0}^{\infty} A_n J_0(\mu_n r) \left[\operatorname{ch} \mu_n (z+\delta) + \frac{\alpha}{\lambda_2 \mu_n} \operatorname{sh} \mu_n (z+\delta) \right]. \tag{9}$$

Here $\mu_n = \gamma_n / R$ $(n = 0, 1, 2 \ldots)$, and the γ_n are the roots of the equation

$$J_1(\gamma_n) = 0. \tag{10}$$

The heat flux recorded by the heat meter is

$$q_H = \frac{\overline{T_2(0)} - \overline{T_2(-\delta)}}{\delta} \lambda_2; \tag{11}$$

$\overline{T}_2(z)$ is the average temperature over the cross section of the heat meter,

$$\overline{T_2(z)} = \frac{1}{\pi R^2} \int_0^R T_2 r dr \int_0^{2\pi} d\varphi = \frac{2}{R^2} \left\{ A_0 \left[1 + \frac{\alpha}{\lambda_2}(z+\delta) \right] \frac{R^2}{2} + \right.$$

$$\left. + \sum_{n=1}^{\infty} A_n \left[\operatorname{ch} \mu_n (z+\delta) + \frac{\alpha}{\lambda_2 \mu_n} \operatorname{sh} \mu_n (z+\delta) \right] \int_0^R J_0(\mu_n r) r dr \right\} = A_0 \left[1 + \frac{\alpha}{\lambda_2} (z+\delta) \right],$$

Since $\int_0^R J_0(\mu_n r) r dr = 0$. Then

$$q_H = A_0 \frac{\lambda_2}{\delta} \left[1 + \frac{\alpha\delta}{\lambda_2} - 1 \right] = \alpha A_0. \tag{13}$$

The value of A_0 can be found by solving the problem for the wall, using boundary conditions (6), (7), and (8). It is easily verified that the general solution of the heat-conduction equation (3) for the wall which is bounded at r = 0 is

$$T_1(r, z) = T_0 \left(1 + \frac{\alpha}{\lambda_1} z \right) + \int_0^{\infty} \exp(-\mu z) J_0(\mu r) f(\mu) d\mu. \tag{14}$$

Here an integral over μ appears instead of a sum over μ_n, since the spectrum of eigenvalues for an infinite region is continuous. Instead of the coefficients A_n there is the unknown function $f(\mu)$. The solution (14) must satisfy the boundary conditions (6), (7), and (8). Since

$$\frac{\partial T_1}{\partial z}(z=0) = \frac{\alpha}{\lambda_1} T_0 - \int_0^\infty \mu J_0(\mu r) f(\mu)\, d\mu;$$

$$T_1(z=0) = T_0 + \int_0^\infty J_0(\mu r) f(\mu)\, d\mu;$$

$$T_2(z=0) = \sum_{n=0}^{\infty} A_n J_0(\mu_n r)\left[\mathrm{ch}(\mu_n\delta) + \frac{\alpha}{\lambda_2\mu_n}\mathrm{sh}(\mu_n\delta)\right];$$

$$\frac{\partial T_2}{\partial z}(z=0) = \sum_{n=0}^{\infty} \mu_n A_n J_0(\mu_n r)\left[\mathrm{sh}(\mu_n\delta) + \frac{\alpha}{\lambda_2\mu_n}\mathrm{ch}(\mu_n\delta)\right],$$

the boundary conditions (6), (7), and (8) can be rewritten in the form

$$\int_0^\infty J_0(\mu r)\left(\frac{\alpha}{\lambda_1} + \mu\right) f(\mu)\, d\mu = 0,\ r > R; \tag{15}$$

$$\int_0^\infty \mu J_0(\mu r) f(\mu)\, d\mu = \frac{\alpha}{\lambda_1} T_0 - \frac{\lambda_2}{\lambda_1}\sum_{n=0}^{\infty} A_n J_0(\mu_n r)\left[\mu_n\,\mathrm{sh}(\mu_n\delta) + \frac{\alpha}{\lambda_2}\mathrm{ch}(\mu_n\delta)\right],\quad r < R; \tag{16}$$

$$\int_0^\infty J_0(\mu r) f(\mu)\, d\mu = -T_0 + \sum_{n=0}^{\infty} A_n J_0(\mu_n r)\left[\mathrm{ch}(\mu_n\delta) + \frac{\alpha}{\lambda_2\mu_n}\mathrm{sh}(\mu_n\delta)\right],\ r < R. \tag{17}$$

Combining (16) and (17) we obtain

$$\int_0^\infty J_0(\mu r)\left(\frac{\alpha}{\lambda_1} + \mu\right) f(\mu)\, d\mu = \frac{\alpha}{\lambda_1}\sum_{n=0}^{\infty} A_n\left(\frac{\alpha}{\lambda_2\mu_n} - \frac{\mu_n\lambda_2}{\alpha}\right) J_0(\mu_n r)\,\mathrm{sh}(\mu_n\delta),\ r < R. \tag{18}$$

From Eqs. (15) and (18) we have

$$\int_0^\infty \varphi(\mu) J_0(\mu r)\, \mu\, d\mu = F(r); \tag{19}$$

$$F(r) = \begin{cases} \sum_{n=0}^{\infty} B_n J_0(\mu_n r); & r < R; \\ 0; & r > R; \end{cases} \tag{20}$$

$$\varphi(\mu) = \frac{\frac{\alpha}{\lambda_1} + \mu}{\mu} f(\mu); \tag{21a}$$

$$B_n = \frac{\alpha}{\lambda_1}\left(\frac{\alpha}{\lambda_2\mu_n} - \frac{\mu_n\lambda_2}{\alpha}\right)\mathrm{sh}(\mu_n\delta)\, A_n; \tag{21b}$$

$$B_0 = \mathrm{Bi}_2 \frac{\alpha}{\lambda_1} A_0\ (\mu_0 = 0).$$

Equation (19) represents the Hankel transform of $\varphi(\mu)$. Using the inversion formula for this transform [4] we obtain

$$\varphi(\mu) = \int_0^\infty F(r) J_0(\mu r) r dr = \sum_{n=0}^\infty B_n \int_0^R J_0(\mu_n r) J_0(\mu r) r dr. \quad (22)$$

Evaluating the integrals of the product of Bessel functions over a finite interval, we have [5]

$$\varphi(\mu) = R\mu J_1(\mu R) \sum_{n=0}^\infty B_n \frac{J_0(\mu_n R)}{\mu^2 - \mu_n^2} = \mathrm{Bi}_1 \mu J_1(\mu R) \sum_{n=0}^\infty A_n \Phi_n,$$

$$\Phi_n = \left(\frac{\alpha}{\lambda_2 \mu_n} - \frac{\mu_n \lambda_2}{\alpha}\right) \mathrm{sh}(\mu_n \delta) \frac{J_0(\mu_n R)}{\mu^2 - \mu_n^2}. \quad (23)$$

This relation expresses the unknown function $\varphi(\mu)$ in terms of the unknown coefficients A_n. However, by using (23) and boundary conditions (16) and (17) a relation among the various coefficients A_n can be obtained. We are interested in only the coefficient A_0, since the heat flux measured by the heat meter is expressed in terms of it. We rewrite boundary condition (16) in the form

$$\int_0^\infty \mu J_0(\mu r) f(\mu) d\mu = \frac{\alpha}{\lambda_1}(T_0 - A_0) - \frac{\lambda_2}{\lambda_1} \sum_{n=1}^\infty A_n J_0(\mu_n r) \left[\mu_n \mathrm{sh}(\mu_n \delta) + \frac{\alpha}{\lambda_2} \mathrm{ch}(\mu_n \delta)\right] \quad (24)$$

and integrate (24) with respect to r from 0 to R using the formula

$$\int_0^R J_0(\mu r) r dr = \frac{R}{\mu} J_1(\mu R).$$

Since $J_1(\mu_n R) = 0$, only the coefficient A_0 remains on the right-hand side of (24) after integration:

$$A_0 = T_0 - \frac{2}{\mathrm{Bi}_1} \int_0^\infty J_1(\mu R) f(\mu) d\mu. \quad (25)$$

Using Eqs. (21a) and (23) for $f(\mu) = \dfrac{\mu\varphi(\mu)}{\dfrac{\alpha}{\lambda_1} + \mu}$, we obtain

$$A_0 = T_0 - 2 \sum_{n=0}^\infty A_n \int_0^\infty \frac{\mu^2}{\dfrac{\alpha}{\lambda_1} + \mu} J_1^2(\mu R) \Phi_n(\mu) d\mu. \quad (26)$$

We multiply both sides of Eq. (24) by $J_0(\mu_n r)$ and integrate with respect to r over the same limits, using the formulas given in [5]:

$$\int_0^R J_0(\mu r) J_0(\mu_n r) r dr = \frac{\mu R}{\mu^2 - \mu_n^2} J_1(\mu R) J_0(\mu_n R);$$

$$\int_0^R J_0(\mu_m r) J_0(\mu_m r) r dr = 0 \quad (\mu_m \neq \mu_n);$$

$$\int_0^R J_0^2(\mu_n r) r dr = \frac{R^2}{2} J_0^2(\mu_n R).$$

We obtain for A_n ($n = 1, 2, \ldots$) the expression

$$A_n = \frac{2}{RJ_0(\mu_n R)\left[\mathrm{ch}(\mu_n\delta) + \frac{\alpha}{\lambda_2\mu_n}\mathrm{sh}(\mu_n\delta)\right]} \times$$

$$\times \int_0^\infty \frac{\mu d\mu}{\mu^2 - \mu_n^2} J_1(\mu R) f(\mu) = \frac{2\mathrm{Bi}_1}{RJ_0(\mu_n R)\left[\mathrm{ch}(\mu_n\delta) + \frac{\alpha}{\lambda_2\mu_n}\mathrm{sh}(\mu_n\delta)\right]} \times$$

$$\times \int_0^\infty \frac{\mu^3 d\mu J_1^2(\mu R)}{(\mu^2 - \mu_n^2)\left(\frac{\alpha}{\lambda_1} + \mu\right)} \sum_{m=0}^\infty A_m \Phi_m(\mu). \quad (27)$$

In integrals (26) and (27) we change to the dimensionless variable $x = \mu R$. Using the notation $\mu_n R = \gamma_n$, where the γ_n are the roots of the equation $J_1(\gamma_n) = 0$, we obtain

$$A_0 = T_0 - 2\sum_{n=0}^\infty A_n \Psi_n \int_0^\infty \frac{x^2 J_1^2(x)\, dx}{(\mathrm{Bi}_1 + x)(x^2 - \gamma_n^2)};$$

$$(28)$$

$$A_n = \frac{2\mathrm{Bi}_1}{J_0(\gamma_n)\left[\mathrm{ch}(\gamma_n\beta) + \frac{Bi_2'}{\gamma_n}\mathrm{sh}(\gamma_n\beta)\right]} \sum_{m=0}^\infty A_m \Psi_m \int_0^\infty \frac{x^3 J_1^2(x)\, dx}{(\mathrm{Bi}_1 + x)(x^2 - \gamma_n^2)(x^2 - \gamma_m^2)} \quad (n = 1, 2, 3, \ldots);$$

$$(29)$$

$$\psi_n = J_0(\gamma_n)\,\mathrm{sh}(\gamma_n\beta)\left(\frac{\mathrm{Bi}_2'}{\gamma_n} - \frac{\gamma_n}{\mathrm{Bi}_2'}\right); \quad \beta = \frac{\delta}{R}.$$

We write

$$A_0 = T_0 - \sum_{n=0}^\infty C_n I_n; \quad (30)$$

$$C_n = \varepsilon a_n \sum_{m=0}^\infty C_m I_{mn}, \quad n \neq 0; \quad (31)$$

$$C_n = \left(\frac{\mathrm{Bi}_2'}{\gamma_n} - \frac{\gamma_n}{\mathrm{Bi}_2'}\right) J_0(\gamma_n)\,\mathrm{sh}(\gamma_n\beta) A_n; \quad (31a)$$

$$C_0 = \mathrm{Bi}_2'\beta A_0 = \mathrm{Bi}_2 A_0; \quad (31b)$$

$$I_n = 2\int_0^\infty \frac{x^2 J_1^2(x)\, dx}{(\mathrm{Bi}_1 + x)(x^2 - \gamma_n^2)}; \quad I_0 = 2\int_0^\infty \frac{J_1^2(x) dx}{\mathrm{Bi}_1 + x};$$

$$I_{mn} = 2\int_0^\infty \frac{x^3 J_1^2(x) dx}{(\mathrm{Bi}_1 + x)(x^2 - \gamma_m^2)(x^2 - \gamma_n^2)};$$

$$I_{0n} = 2\int_0^\infty \frac{x J_1^2(x)\, dx}{(\mathrm{Bi}_1 + x)(x^2 - \gamma_n^2)}; \quad (32)$$

$$\varepsilon = \frac{\mathrm{Bi}_1}{\mathrm{Bi}_2'} = \frac{\lambda_2}{\lambda_1};\quad a_n = \frac{\mathrm{Bi}_2'^2 - \gamma_n^2}{\mathrm{Bi}_2' + \gamma_n \operatorname{cth}(\gamma_n\beta)}. \tag{33}$$

The integrals for I_n and I_{mn} have no singularities at $x = \gamma_n$ or $x = \gamma_m$, since as $x \to \gamma_n$, $J_1(x) \sim x^2 - \gamma_n^2$ [5]. Equations (28), (29) or (30), (31) form an infinite system of algebraic equations. The theory of infinite systems of equations is not yet in final form, but methods for obtaining approximate solutions of such equations are developed in considerable detail in [6]. In general these methods lead to lengthy cumbersome calculations, but if there is a small parameter in the problem, the solution of such a system can be written as a series in this parameter.

We solve the problem by assuming that the coefficient ε in Eq. (31) is such a small parameter. We transform Eqs. (30) and (31) to a form convenient for iterations:

$$A_0 = T_0 - C_0 I_0 - \sum_{n=1}^{\infty} C_n I_n, \tag{34}$$

$$C_n = \varepsilon a_n C_0 I_{0n} + \varepsilon a_n \sum_{m=1}^{\infty} C_m I_{mn},\quad n \neq 0. \tag{35}$$

Iterating Eq. (35) we obtain

$$C_n = \varepsilon a_n C_0 I_{0n} + \varepsilon a_n \varepsilon C_0 \sum_{m=1}^{\infty} a_m I_{0m} I_{mn} + \varepsilon a_n \sum_{m=1}^{\infty} \varepsilon a_m \varepsilon C_0 \left(\sum_{k=1}^{\infty} a_k I_{0k} I_{km}\right) I_{mn} + \ldots = \varepsilon C_0 a_n F_n; \tag{36}$$

$$F_n = I_{0n} + \varepsilon \sum_{m=1}^{\infty} a_m I_{0m} I_{mn} + \varepsilon^2 \sum_{m,k} a_m a_k I_{0k} I_{km} I_{mn} + \ldots \tag{36a}$$

Substituting (36) into (34) and using (36b), we obtain

$$A_0 = T_0 - C_0 I_0 - \varepsilon C_0 \sum_{n=1}^{\infty} a_n F_n I_n = T_0 - \mathrm{Bi}_2 A_0 \,[I_0 + \varepsilon\Psi]; \tag{37}$$

$$\Psi = \sum_{n=1}^{\infty} a_n F_n I_n;\quad A_0\,[1 + \mathrm{Bi}_2 (I_0 + \varepsilon\Psi)] = T_0. \tag{38}$$

The distorted heat flux indicated by the heat meter is $q_H = \alpha A_0$, and the heat flux in the absence of the heat meter (undistorted heat flux) is $q_0 = \alpha T_0$. Therefore, Eq. (38) relates the value of the undistorted heat flux to the reading of the heat meter and provides the necessary correction to be applied to the instrument reading to obtain the value of the undistorted flux:

$$q_0 = q_H [1 + \mathrm{Bi}_2 (I_0 + \varepsilon\Psi)]. \tag{39}$$

Thus, the solution is given in the form of an infinite sum, but these sums converge rather rapidly. Actually

$$\Psi = \sum_{n=1}^{\infty} a_n I_n F_n \approx \sum_{n=1}^{\infty} a_n I_n I_{0n}.$$

For large γ_n the quantities $a_n \simeq -\gamma_n$, $I_n \sim 1/\gamma_n^2$, and $I_{0n} \sim 1/\gamma_n^2$, and therefore the n-th term of the series is of order $1/\gamma_n^3$. Actually it turns out to be sufficient to retain only two or three terms of the series. Taking only two terms approximates the sum to within 5%.

NOTATION

q_0, unperturbed heat flux; q_H, heat flux recorded by heat meter; α, heat-transfer coefficient; λ, thermal conductivity; δ, thickness of heat meter; R, radius of heat meter; $\beta = \delta/R$; $\mathrm{Bi}_1 = \alpha R/\lambda_1$; $\mathrm{Bi}_2 = \alpha\delta/\lambda_2$; $\mathrm{Bi}_2' = \alpha R/\lambda_2$; J_0, J_1, zero- and first-order Bessel functions, respectively; T_1, T_2, temperature distributions of semi-infinite wall and heat meter, respectively; T_0, temperature distribution of wall in the absence of heat meter.

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MASS TRANSFER IN EVACUATION OF MATERIALS WITH LARGE OUTGASSING

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Non-steady-state mass transfer in materials with large outgassing, used for the heat insulation of cryogenic vessels, was investigated experimentally and theoretically.

Materials used for vacuum-multilayered insulation (VMI) of cryogenic vessels have an extended surface and, consequently, large outgassing in vacuum. The required vacuum in the insulation cavity of the vessels is maintained with the aid of adsorbents which also liberate a large amount of gases during the initial evacuation. All this leads to a substantial extension of the time of evacuation, which sometimes attains more than 100 h.

The outgassing of various materials for VMI was measured by the authors of [1-4]. The experimental data obtained by different authors for the same materials differ, sometimes one being a multiple of the other. The object of the present work is to find the causes of these discrepancies and to work out a method of calculating the process of evacuation on the basis of its theoretical and experimental investigation.

The equation of non-steady-state mass transfer in the diffusion of a sorbed gas with a linear adsorption isotherm in a plane layer of porous material has the form [5]

$$\frac{\partial c}{\partial \tau} = D_e \frac{\partial^2 c}{\partial x^2}. \qquad (1)$$

The effective diffusion coefficient $D_e = D/(1 + H)$, when H is the Henry law constant characterizing the slope of the adsorption isotherm, is

$$da = H\,dc. \qquad (2)$$

The absorption per unit mass of the sorbent is

$$da = \frac{H}{\rho}\,dc. \qquad (2')$$

Since the gas pressure is proportional to the concentration, Eq. (1) may be replaced by

$$\frac{\partial p}{\partial \tau} = D_e \frac{\partial^2 p}{\partial x^2}. \qquad (3)$$

Some authors (e.g., Mikhal'chenko and Pershin [6]) veiw the process of evacuation of the insulation as a diffusion process with distributed sources of outgassing, with the

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